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INEXTENSIONAL BENDING IN SHELLS OF EXPLICIT CUBIC REPRESENTATION†

L. S. D. MORLEY
Materials/Structures Department, Royal Aircraft Establishment, Farnborough, Hampshire, England

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Abstract—Formulae are derived for exact inextensional bending solutions of shell surfaces of explicit cubic representation. The illustrative example treats a shell surface which has the same shape as a uniformly loaded soap bubble stretched from an equilateral triangular boundary.

1. INTRODUCTION

Exact solutions have been recently derived[1] where the small displacement inextensional bending of shell surfaces of explicit quadratic representation $Z = a_1 + a_2X + a_3Y + \cdots a_6Y^2$ is expressed by rectangular displacement components U_X, U_Y, U_Z which are simple polynomials of the rectangular coordinates X and Y. These solutions are helpful in shell finite element evaluation.

This class of shell surfaces is now widened to explicit cubic representation $Z = a_1 + a_2X + a_3Y + \cdots a_{10}Y^3$ and it is shown that this is, generally, the highest degree of shell surface polynomial representation which admits inextensional bending solutions of this kind.

Previous work[1] provides details, not repeated here, of supporting analysis as well as discussing the rôle of inextensional bending especially with reference to the finite element method. Attention is, however drawn to Gol'denveizer's[2] celebrated static-geometric analogue which discloses that membrane states satisfying the differential equations of equilibrium with zero surface forces are the analogues of inextensional bending states.

Example solutions are given of inextensional bending deformations of a shell surface which has the same shape as a uniformly loaded soap bubble stretched from an equilateral triangular boundary.

2. GENERAL METHOD OF SOLUTION

Let X, Y, Z refer to a fixed right handed rectangular coordinate system with the shell middle surface explicitly defined by

$$Z = Z(X, Y) \tag{2.1}$$

over a region where derivatives of Z exist.

The projections of X, Y onto the shell middle surface trace out a curvilinear coordinate system which is no longer orthogonal. The strain in this surface is, however, completely prescribed by curvilinear strain components ϵ_{XX}, ϵ_{YY}, ϵ_{XY} where

$$\begin{aligned} \epsilon_{XX} &= \frac{1}{\alpha_X^2}\left(\frac{\partial U_X}{\partial X} + \frac{\partial Z}{\partial X}\frac{\partial U_Z}{\partial X}\right), \\ \epsilon_{YY} &= \frac{1}{\alpha_Y^2}\left(\frac{\partial U_Y}{\partial Y} + \frac{\partial Z}{\partial Y}\frac{\partial U_Z}{\partial Y}\right), \\ \epsilon_{XY} &= \frac{1}{2\alpha_X\alpha_Y}\left(\frac{\partial U_X}{\partial Y} + \frac{\partial U_Y}{\partial X} + \frac{\partial Z}{\partial Y}\frac{\partial U_Z}{\partial X} + \frac{\partial Z}{\partial X}\frac{\partial U_Z}{\partial Y}\right) \end{aligned} \tag{2.2}$$

with

$$\alpha_X^2 = 1 + \left(\frac{\partial Z}{\partial X}\right)^2, \quad \alpha_Y^2 = 1 + \left(\frac{\partial Z}{\partial Y}\right)^2 \tag{2.3}$$

the coefficients of the first fundamental form. Note that U_X, U_Y, U_Z are displacement components in the rectangular coordinate system. These curvilinear strain components satisfy the following differential relationship

$$\frac{\partial^2}{\partial Y^2}(\alpha_X^2\epsilon_{XX}) + \frac{\partial}{\partial X^2}(\alpha_Y^2\epsilon_{YY}) - 2\frac{\partial^2}{\partial X\partial Y}(\alpha_X\alpha_Y\epsilon_{XY}) = 2\frac{\partial^2 Z}{\partial X\partial Y}\frac{\partial^2 U_Z}{\partial X\partial Y} - \frac{\partial^2 Z}{\partial Y_2}\frac{\partial^2 U_Z}{\partial X^2} - \frac{\partial^2 Z}{\partial X^2}\frac{\partial^2 U_Z}{\partial Y^2}. \tag{2.4}$$

Consequently, each solution for U_Z to the homogeneous differential equation

$$2\frac{\partial^2 Z}{\partial X\partial Y}\frac{\partial^2 U_Z}{\partial X\partial Y} - \frac{\partial^2 Z}{\partial Y^2}\frac{\partial^2 U_Z}{\partial X^2} - \frac{\partial^2 Z}{\partial X^2}\frac{\partial^2 U_Z}{\partial Y^2} = 0 \tag{2.5}$$

determines an inextensional bending deformation of the explicitly defined shell surface where $\epsilon_{XX} = \epsilon_{YY} = 0$ provides

$$\left.\begin{aligned} U_X &= -\int_0^X \frac{\partial Z}{\partial X}\frac{\partial U_Z}{\partial X}\,\mathrm{d}X + F(Y), \\ U_Y &= -\int_0^Y \frac{\partial Z}{\partial Y}\frac{\partial U_Z}{\partial Y}\,\mathrm{d}Y + G(X), \end{aligned}\right\} \tag{2.6}$$

whereupon considering $\epsilon_{XY} = 0$ along $X = 0$ and then along $Y = 0$ provides

$$\left.\begin{aligned} \frac{\partial F}{\partial Y} &= \left[2\frac{\partial^2 Z}{\partial X\partial Y}U_Z - \frac{\partial}{\partial X}\int_0^Y \frac{\partial^2 Z}{\partial Y^2}U_Z\,\mathrm{d}Y\right]_{X=0}, \\ \frac{\partial G}{\partial X} &= \left[2\frac{\partial^2 Z}{\partial X\partial Y}U_Z - \frac{\partial}{\partial Y}\int_0^X \frac{\partial^2 Z}{\partial X^2}U_Z\,\mathrm{d}X\right]_{Y=0} \end{aligned}\right\} \tag{2.7}$$

Interest in the finite element method focuses attention onto solutions of eqn (2.5) where Z is polynomial of degree $p \geqslant 2$ in X and Y. Here, let there be polynomial solutiosn U_Z of degree $q \geqslant 2$. Existence generally requires that

$$(q+1)(q+2) - 6 \equiv (q-1)(q+4) > (q+p-3)(q+p-2). \tag{2.8}$$

For Z quadratic, $p = 2$, this gives $q \geqslant 2$ as is already known[1]. For Z cubic, $p = 3$, then $q \geqslant 3$ and this receives attention in the sequel. Equation (2.8) cannot be satisfied for integer $p \geqslant 4$ and it is therefore generally impossible then to find exact inextensional bending solutions of this kind. There are, however special situations when eqn (2.8) does not apply as may be illustrated by considering consequences of the analogy

$$U_Z \rightarrow Z, \quad Z \rightarrow U_Z. \tag{2.9}$$

The differential operator $L(Z, U_Z)$, implicit in eqns (2.4) and (2.5), occurs frequently in plate and shell theory, it is sometimes called Pucher's[3, 4] operator. Golden'veizer's static-geometric analogue anticipates occurrence of the operator when calculating membrane stresses in shells, see Flügge[5]. In earlier work Flügge and Geyling[6], following Pucher's approach, explicitly derive eqn (2.4) when treating displacements in membrane shells. Mansfield[7, 8] provides interesting relationships for the differential operator in the course of work in small and large deflection plate theory.

3. SHELL SURFACES OF EXPLICIT CUBIC REPRESENTATION

Consideration is now given to the inextensional bending of shell surfaces of explicit cubic representation,

$$Z = a_1 + a_2X + a_3Y + a_4X^2 + a_5XY + a_6Y^2 + a_7X^3 + a_8X^2Y + a_9XY^2 + a_{10}Y^3. \quad (3.1)$$

The first $n - 3$ solutions to eqn (2.5) may then be written

$$U_Z = \sum_{i=3}^{n} \sum_{j=1}^{i} c_{k(i,j)} X^{i-j} Y^{j-1} \quad (3.2)$$

with

$$k(i,j) = \tfrac{1}{2}i(i-1) + j - 3 \quad (3.3)$$

and $n > 3$. The $\frac{1}{2}n(n+1) - 3$ constants c_k are subject to $\frac{1}{2}n(n-1)$ conditions in the course of satisfying eqn (2.5) but are otherwise arbitrary.

Displacement components U_X, U_Y are readily calculable from eqns (2.6) with

$$\left.\begin{aligned} \frac{\partial F}{\partial Y} &= 2\sum_{i=3}^{n}\left\{\left(a_5 + \frac{2i-1}{i}a_9Y\right)c_{k(i,i)} - \left(\frac{1}{i-1}a_6 + \frac{3}{i}a_{10}Y\right)c_{k(i,i-1)}\right\}Y^{i-1}, \\ \frac{\partial G}{\partial X} &= 2\sum_{i=3}^{n}\left\{\left(a_5 + \frac{2i-1}{i}a_8X\right)c_{k(i,1)} - \left(\frac{1}{i-1}a_4 + \frac{3}{i}a_7X\right)c_{k(i,2)}\right\}X^{i-1}. \end{aligned}\right\} \quad (3.4)$$

4. EXAMPLE

By way of illustration, algebraic expressions are written out for the first two inextensional bending deformations of a shell surface which has the same shape as a uniformly loaded soap bubble stretched from the equilateral triangular boundary shown in Fig. 1. This surface is described by

$$Z = \frac{4}{27} - X^2 - Y^2 + X^3 - 3XY^2 \quad (4.1)$$

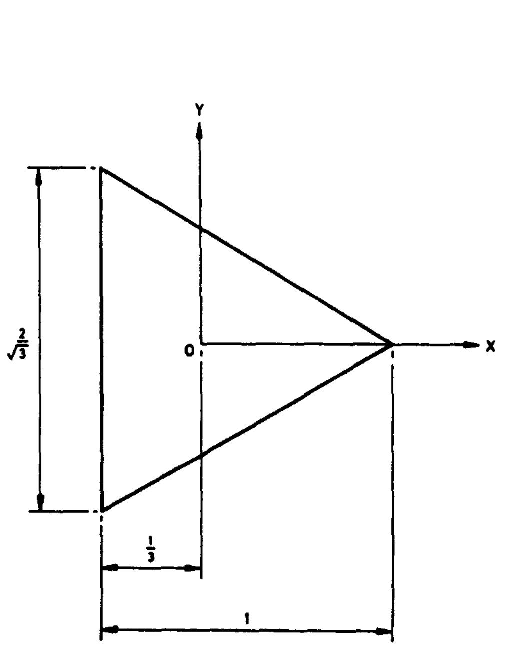

Fig. 1. Equilateral triangular boundary at $Z = 0$.

so that

$$\left.\begin{array}{ll} a_1 = \dfrac{4}{27}, & a_6 = -1, \\ a_2 = 0, & a_7 = 1, \\ a_3 = 0, & a_8 = 0, \\ a_4 = -1, & a_9 = -3, \\ a_5 = 0, & a_{10} = 0. \end{array}\right\} \tag{4.2}$$

For the first inextensional bending deformation the integer $n = 4$ in eqn (3.2) so that

$$U_Z = c_1X^2 + c_2XY + c_3Y^2 + c_4X^3 + c_5X^2Y + c_6XY^2 + c_7Y^3. \tag{4.3}$$

Satisfaction of eqn (2.5) demands

$$Ac = 0 \tag{4.4}$$

where the 6 by 7 matrix A is

$$A = \begin{bmatrix} 4 & & 4 & & & & \\ 12 & & -12 & 12 & & 4 & \\ & -12 & & & 4 & & 12 \\ & & & 36 & & -12 & \\ & & & & -12 & & -36 \\ & & & & & -24 & \end{bmatrix} \tag{4.5}$$

and

$$c^T = (c_1\, c_2\, c_3\, c_4\, c_5\, c_6\, c_7). \tag{4.6}$$

The eigenvector of A^TA which has zero eigenvalue is available from inspection

$$c^T = \left(0\,0\,0\,0\,1\,0\,-\frac{1}{3}\right) \tag{4.7}$$

and this gives

$$U_Z = X^2Y - \frac{1}{3}Y^3. \tag{4.8}$$

The functions $F(Y)$ and $G(X)$ are readily determined from eqns (2.7),

$$F = \frac{7}{10}Y^5, \quad G = \frac{1}{6}X^4 - \frac{3}{10}X^5, \tag{4.9}$$

so that the displacement components U_X and U_Y of eqns (2.6) are

$$\left.\begin{array}{l} U_X = \dfrac{4}{3}X^3Y - \dfrac{3}{2}X^4Y + 3X^2Y^3 - \dfrac{3}{10}Y^5, \\ U_Y = \dfrac{1}{6}X^4 + X^2Y^2 - \dfrac{1}{2}Y^4 - \dfrac{3}{10}X^5 + 3X^3Y^2 - \dfrac{3}{2}XY^4. \end{array}\right\} \tag{4.10}$$

Substitution of eqns (4.8) and (4.10) into eqn (2.2) confirms that the curvilinear strain components ϵ_{XX}, ϵ_{YY}, ϵ_{XY} are zero as is required for inextensional bending.

For the second inextensional bending deformation the integer $n = 5$ in eqn (3.2) and

$$U_Z = c_1X^2 + c_2XY + c_3Y^2 + c_4X^3 + c_5X^2Y + c_6XY^2 + c_7Y^3 + c_8X^4 + c_9X^3Y + c_{10}X^2Y^2 + c_{11}XY^3 + c_{12}Y^4. \tag{4.11}$$

The 10 by 12 matrix A, eqn (4.4), is now

$$A = \begin{bmatrix} 4 & & 4 & & & & & & & & & \\ 12 & & -12 & 12 & & 4 & & & & & & \\ & -12 & & & 4 & & 12 & & & & & \\ & & & 36 & & -12 & & 24 & & 4 & & \\ & & & & -12 & & -36 & & 12 & & 12 & \\ & & & & & -24 & & & & 4 & & 24 \\ & & & & & & & 72 & & -12 & & \\ & & & & & & & & & & -36 & \\ & & & & & & & & & -36 & & -72 \\ & & & & & & & & & & -36 & \end{bmatrix} \tag{4.12}$$

with

$$c^T = (c_1\, c_2\, c_3 \ldots c_{12}). \tag{4.13}$$

Exceptionally this A^TA has three (rather than two) zero eigenvalues, the first eigenvector repeating eqn (4.7) while the second and third eigenvectors provide identical inextensional bending deformations on rotating the triangle. The significant eigenvector gives

$$U_Z = \frac{1}{3}XY + X^2Y + X^3Y. \tag{4.14}$$

The functions $F(Y)$ and $G(X)$ are

$$F = \frac{1}{9}Y^3, \quad G = \frac{1}{9}X^3 - \frac{1}{5}X^5 - \frac{1}{5}X^6, \tag{4.15}$$

and the remaining displacement components are

$$\left.\begin{aligned} U_X &= \frac{1}{3}X^2Y + \frac{1}{9}Y^3 + X^3Y + XY^3 + 3X^2Y^3 - \frac{9}{5}X^5Y + 3X^3Y^3, \\ U_Y &= \frac{1}{9}X^3 + \frac{1}{3}XY^2 + 2X^2Y^2 - \frac{1}{5}X^5 + 4X^3Y^2 - \frac{1}{5}X^6 + 3X^4Y^2. \end{aligned}\right\} \tag{4.16}$$

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